Engineering Notes

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The Closed-Form Solution of Generalized Proportional Navigation

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Introduction

PURE proportional navigation (PPN)[1] and true proportional navigation (TPN)[2] have been widely used for homing missile guidance. In PPN, the commanded missile acceleration is applied normal to the missile velocity; in TPN, the missile acceleration is applied normal to the line of sight (LOS). In this Note, generalized proportional navigation (GPN) is defined as similar to proportional navigation, except that the missile acceleration is not necessarily applied normal to LOS. A closed-form solution of the equations of motion of an ideal missile pursuing a nonmaneuvering target under GPN is developed. Several properties of GPN are described in this Note and it is shown that, under certain conditions, GPN has a larger capture area and a shorter interception time than TPN.

Analysis

To derive the equation of motion, let V be the target vector velocity relative to the interceptor. In polar coordinates (r,θ) with the origin fixed at the interceptor, V is

$$V=V_r e_r + V_\theta e_\theta = \dot{r} e_r + r\dot{\theta} e_\theta \tag{1}$$

From elementary kinematics, we have

$$\mathrm{d}V/\mathrm{d}t = a_t - a_m \tag{2}$$

where a_t is the target acceleration (which is zero for a nonmaneuvering target) and a_m the interceptor acceleration.

Define ψ as the angle between the direction of the interceptor acceleration and the direction normal to the LOS. In GPN, the interceptor acceleration is proportional to the LOS angular rate as

$$a_m = k_r \dot{\theta} e_r + k_\theta \dot{\theta} e_\theta \tag{3}$$

where $k_r = \lambda V_{\theta_0} \sin(\psi)$, $k_\theta = -\lambda V_{r_0} \cos(\psi)$, and λ is the navigation constant.

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Substituting for V and a_m from Eqs. (1) and (3), Eq. (2) yields

$$\dot{V}_r = (V_\theta - k_r)\dot{\theta} \tag{4a}$$

$$\dot{V}_\theta = -(V_r + k_\theta)\dot{\theta} \tag{4b}$$

or, in terms of r and θ,

$$\ddot{r} - r\dot{\theta}^2 = -k_r \dot{\theta} \tag{5a}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -k_\theta \dot{\theta} \tag{5b}$$

For the special case $k_r = 0$, which corresponds to TPN, the closed-form solution of Eqs. (4) has been obtained by Guelman.[3] However, his method cannot be extended to the general case of $k_r \neq 0$. The alternate approach given here provides the trajectories of the missile in relative coordinates, according to GPN but with conditions under which GPN is found to be superior to TPN.

By changing the independent variable from t to θ, Eqs. (4) are reduced to the form

$$\mathrm{d}V_r/\mathrm{d}\theta = V_\theta - k_r \tag{6a}$$

$$\mathrm{d}V_\theta/\mathrm{d}\theta = -V_r - k_\theta \tag{6b}$$

The solution of this set of ordinary differential equations with initial conditions

$$V_r(\theta_0) = V_{r_0} \quad \text{and} \quad V_\theta(\theta_0) = V_{\theta_0}$$

is found to be

$$-V_r = R\cos(\phi) + k_\theta \tag{7a}$$

$$V_\theta = R\sin(\phi) + k_r \tag{7b}$$

where

$$R = [(V_{r_0} + k_\theta)^2 + (V_{\theta_0} - k_r)^2]^{1/2}$$

$$\phi = \theta - \theta_0 + \alpha$$

$$\tan(\alpha) = -(V_{\theta_0} - k_r)/(V_{r_0} + k_\theta)$$

Equations (7) show that the hodograph is a circle with center (k_r, k_θ) and radius R. If we choose $\theta_0 = \alpha$, then the intersection between the LOS and the hodograph uniquely determines the velocity components V_r and V_θ.

To make Eqs. (7) dimensionless, divide Eqs. (7a) and (7b) by V_{r_0} and V_{θ_0}, respectively. Then,

$$u = a\cos(\phi) + \lambda_1 \tag{8a}$$

$$v = b\sin(\phi) + \lambda_2 \tag{8b}$$

for $V_{\theta_0} > 0$. If $V_{\theta_0} < 0$, replace ϕ by $-\phi$. The dimensionless variables are defined as

$$u = V_r / V_{r_0}, \quad v = V_\theta / V_{\theta_0}$$

$$b = [c^2(1-\lambda_1)^2 + (1-\lambda_2)^2]^{1/2}, \quad c = |V_{r_0}/V_{\theta_0}|$$

$$a = b/c, \quad \lambda_1 = \lambda \cos(\psi), \quad \lambda_2 = \lambda \sin(\psi)$$

Defining the angular momentum per unit mass in the relative coordinate system to be

$$H = rV_\theta$$

and the corresponding nondimensional form to be

$$h = H/rV_{\theta_0} = H/H_0$$

then Eqs. (5b) can be rewritten as

$$\frac{\mathrm{d}h}{\mathrm{d}\tau} = \frac{Kh\lambda_1}{x} \tag{9}$$

where $x = r/r_0$, $\tau = t/t_f$, $K = V_{r_0}$, and t_f is the final time or end of the pursuit. An important equation relating h to aspect angle θ can be established by combining Eqs. (1), (8), and (9). After some manipulation, we have

$$-\frac{\lambda_1 b/a}{b \sin(\phi) + \lambda_2} \mathrm{d}\phi = \frac{\mathrm{d}h}{h} \tag{10}$$

According to the value of ψ, there are four separate cases to consider,

$$1)\ \lambda_2 = 0, \quad 2)\ 0 < \lambda_2 < b, \quad 3)\ \lambda_2 = b, \quad 4)\ b < \lambda_2$$

The integration for each case can be found in standard mathematical tables. Once the relation between ϕ and h has been established, the other quantities can be obtained immediately. It has been shown[4] that the above approach gives the same results as Guelman's solution for case 1, i.e., for TPN. Below we give part of the results for case 2, which reveal some interesting properties of GPN. For detailed derivations and the corresponding geometrical interpretation, see Ref. 4.

For case 2, the integration of Eq. (10) gives

$$\frac{\lambda_2 \tan(\phi/2) + b - (b^2 - \lambda_2^2)^{1/2}}{\lambda_2 \tan(\phi/2) + b + (b^2 - \lambda_2^2)^{1/2}} = Ah^{-m} \tag{11}$$

where A is the initial value of the left-hand side term and

$$m = a(b^2 - \lambda_2^2)^{1/2} / \lambda_1 b$$

From Eqs. (8), (9), and (11), h is related to τ as

$$\frac{bA}{1-m}[b + (b^2 - \lambda_2^2)^{1/2}]h^{1-m} - 2\lambda_2 h + \frac{b}{A(1+m)}[b - (b^2 - \lambda_2^2)^{1/2}]h^{1+m} = 2K\lambda_1\lambda_2(b^2 - \lambda_2^2)\tau + C_\tau \tag{12}$$

where C_τ is determined from the initial condition

$$h = 1 \quad \text{at} \quad \tau = 0$$

The nondimensional relative distance is found to be

$$x = \{b[b + (b^2 - \lambda_2^2)^{1/2}]h^{1-m} - 2\lambda_2^2 h + b[b - (b^2 - \lambda_2^2)^{1/2}]h^{1+m}\}/2\lambda_2(b^2 - \lambda_2^2) \tag{13}$$

For the missile to reach the target at a finite time, we must have from Eqs. (12) and (13) that

$$m < 1$$

or, equivalently,

$$c^2 < -\frac{1 - 2\lambda \sin(\psi)}{1 - 2\lambda \cos(\psi)} \tag{14}$$

This is depicted in Fig. 1. The trajectory of the missile is restricted to the right-hand region of each curve representing the different values of ψ. The total capture area, according to the GPN law, is characterized by the right-hand region of the envelope and denoted by

$$\lambda = (c^2 + 1)/2(c^4 + 1)^{1/2} \tag{15}$$

It is obvious from Fig. 1 that GPN has a larger capture area than TPN ($\psi = 0$) in the region where c is small.

The final time or end of the pursuit can be found from Eqs. (12) with the terminal condition $h = 0$ at $\tau = 1$. The result is illustrated in Fig. 2 for $c = 0.5$. A remarkable property of GPN is that the final time is very sensitive to variations of ψ when c and ψ are small. Thus, the interception time can be greatly

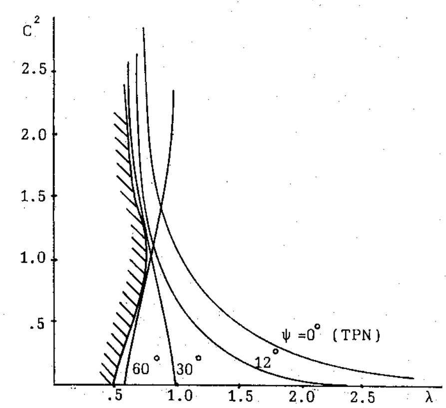

Fig. 1 Capture area for TPN and GPN.

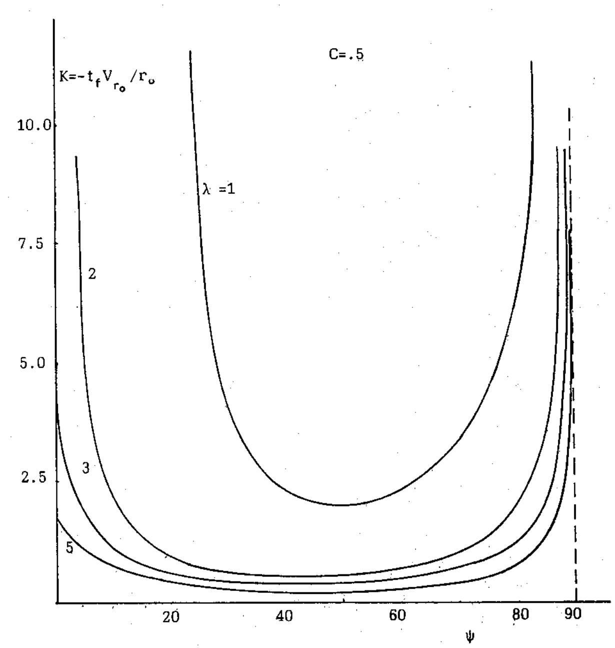

Fig. 2 Final time of pursuit termination ψ for various proportional constants.

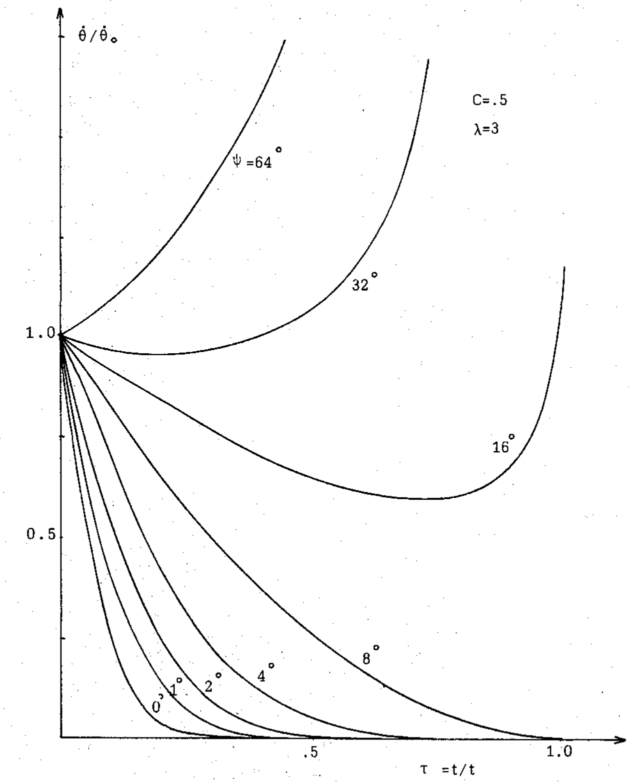

Fig. 3 Line-of-sight rotational rate vs time for various values of ψ.

reduced if ψ is increased from zero to a small positive value for a small value of c. The LOS angular rate $\dot{\theta}=H/r$ can be obtained from Eq. (13). The result is depicted in Fig. 3, where it can be seen that a large value of ψ results in an unbounded missile acceleration. It can also be shown[4] that the missile acceleration is always finite at the end of the pursuit if $m \leq ½$.

From the several results considered above, it is suggested that GPN be used under conditions where c is small. A small value of c represents a small closing rate or a large target tangential velocity relative to the interceptor. Under these conditions, GPN has a larger capture area and a shorter interception time than TPN and the increment of the acceleration from that of TPN is small when ψ is kept small.

Conclusion

In this study, we have derived the closed-form solution of the differential equations describing the trajectories of a missile pursuing a nonmaneuvering target according to the generalized proportional navigation (GPN) law. Of great interest in this study is the fact that the analysis of the closed-form solution of GPN has enabled us to demonstrate the effectiveness of the concept.

A proper choice of ψ, the angle between the direction of interceptor acceleration and the direction normal to the line of sight, involves a tradeoff between the interceptor acceleration commanded and the interception time. Instead of a constant value of ψ, an optimal trajectory of ψ can be found if the weighted sum of these two factors is minimized.

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A New Method of Computing the State Transition Matrix for Linear Systems

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Introduction

A NEW method of computing the state transition matrix for a linear, time-invariant system is presented in this Note. Such a computation is required in estimation, filtering, and control problems where predictive computations are necessary.

Let a linear, time-invariant system be represented by

$$\dot{x}(t)=Fx(t)+Gu(t), \qquad t\epsilon[t_0,t_f] \tag{1}$$

with initial condition $x(t_0)=x_0$, where the n-vector x denotes the state of the system, the p-vector u a forcing function, F and G the $n\times n$ continuous dynamics and the $n\times p$ forcing coupling matrices, respectively.

The solution[1,2] of Eq. (1) is obtained by means of the state transition matrix (or the fundamental matrix) $\Phi(t,t_0)$ for the homogeneous linear system corresponding to Eq. (1). For such a system, the transition matrix is just the exponential of the dynamics matrix F, the Peano-Baker formula.[3] The technique for computing the state transition matrix presented in this Note may be considered to be a new way of computing the exponential of a constant matrix. The problem of exponentiating a constant matrix has been studied extensively. Moler and van Loan[4] discuss at length 19 "dubious" ways of computing the exponential of a constant matrix. Reference 5 addresses the same problem and contains a good list of papers on various techniques for computing the exponential of a matrix.

Some discussion of the attributes of our technique is in order. It does not require a priori knowledge of the eigenvalues and eigenvectors of the dynamics (system) matrix. It is effective for a wide class of system matrices. It takes full advantage of the linear time-invariant nature of the system. Large transition intervals need not be divided into small transition intervals in order to preserve desired accuracy; our technique is valid for any finite transition interval.

We derive herein a compact formula for Φ for a consant matrix F, using what is known as the integral variation (IV) method.[6] We then employ this formula, the main result of this Note, to implement the prediction mode of a Kalman filter used for estimating the state (position and velocity) of a damped oscillator.

A Compact Formula for the Transition Matrix

Our method involves solving the homogeneous system corresponding to Eq. (1) and then obtaining the state transition matrix from the resulting solution. Hereafter, all vectors are column vectors.

Briefly, the IV method is a technique to generate an approximate solution of an initial value problem, whose original differential equations have been transformed into a system of

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